Quantifying Patterns and Their Transitions in Spatially Extended Systems

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Overview

Goal: Develop a general framework and a corresponding continuation algorithm to identify curves in 2D parameter space that distinguish dynamically different patterns.

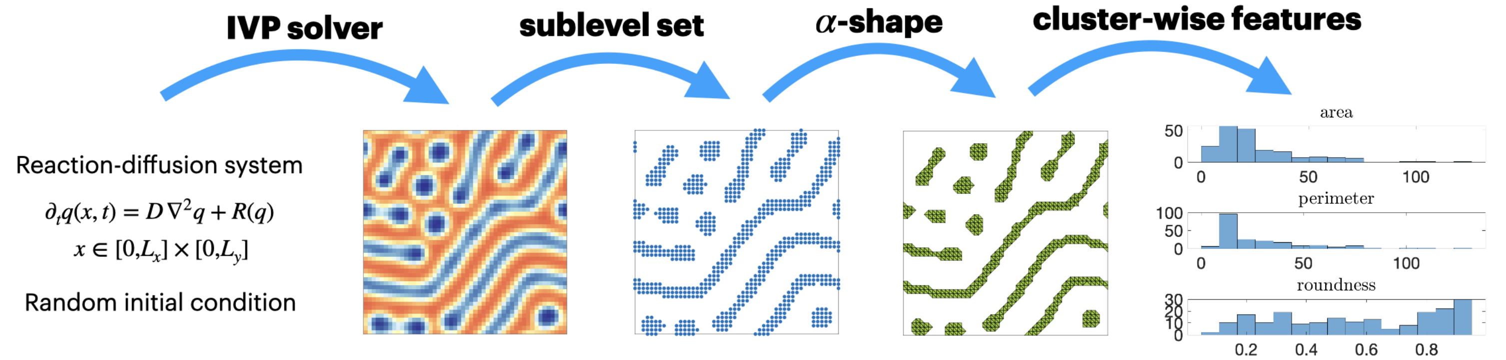
Our approach:

- Input: Initial-value problem solvers with random initial conditions;
- Feature: Number/roundness of α-shape geometry;
- Metric: Wasserstein distance defined in feature space;
- Continuation: Find maxima in parameter space.

Novelty: Purely data-driven, allowing for automatic and efficient bifurcation tracing with limited prior knowledge of the underlying system.

Application: Snaking, homogeneous states, spots, stripes, and spiral waves.

Pattern statistics via features



Comparison of patterns and continuation algorithm

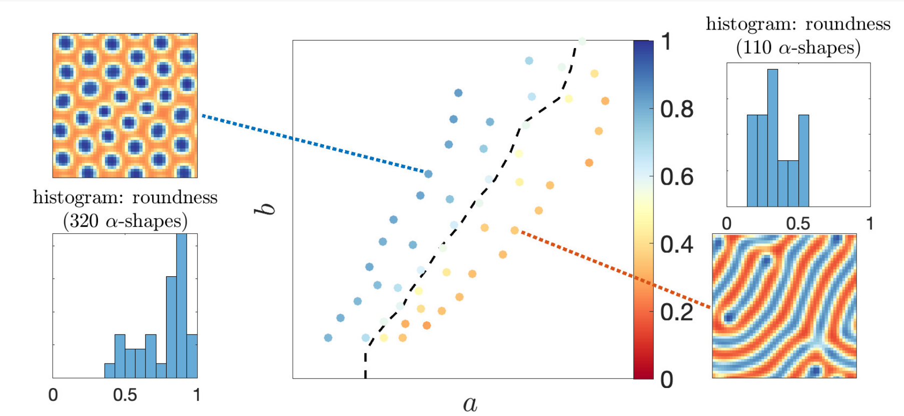


Figure: Bifurcation curve in 2D parameter space (example: Brusselator). Color code: average roundness of α-shapes. To quantify difference between patterns, we use Wassterstein distance between histograms of pattern statistics.

Our pseudo-arclength continuation algorithm:

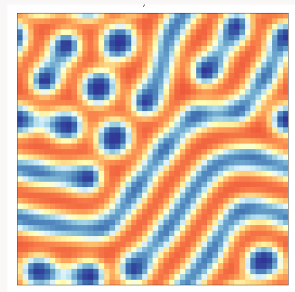
In each step:

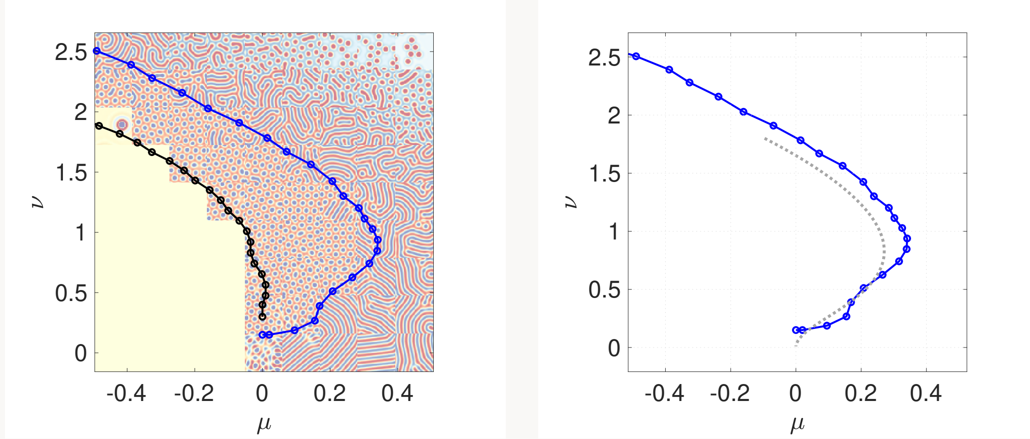
- Identify interval to investigate based on initial parameters/directions;
- Generate simulations and compute Wasserstein distance in feature space $f(p) = d_{feature}(p+\Delta p, p-\Delta p)$;
- Find maxima of $f(p)$ via quadratic approximation;
- Update parameter and direction for next step.

Example 1: Turing patterns in 2D Swift-Hohenberg model [2]

Features: number/roundness of α-shapes constructed based on sublevel sets of PDE solution.

Turing patterns (spots and stripes)





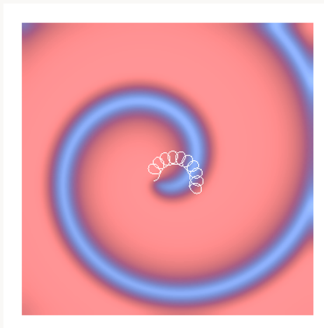
(a) Comparison to simulation (b) Comparison to reference

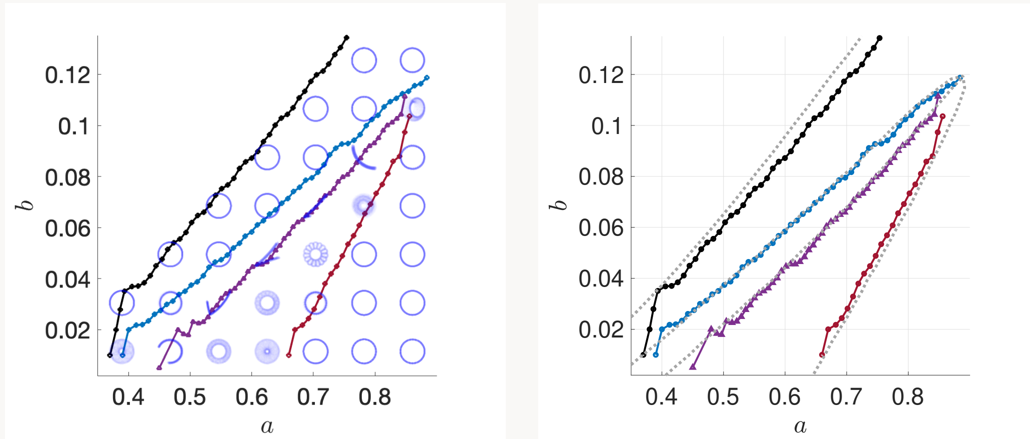
Figure: Curves separating (1) homogeneous vs spots (black); (2) spots vs stripes (blue), compared to simulation and Maxwell curve.

Example 2: Tip motion of spiral waves in Barkley model [3]

Features: thickness/curvature of α-shapes constructed based on tip trajectory.

spiral wave (blue/red) and tip motion (white)





(a) Comparison to simulation (b) Comparison to reference

Figure: Curves separating (1) homogeneous vs rigid rotation (black); (2) rigid rotation vs meander (blue and red); (3) meander with different directions (purple), compared to simulations/references.

References

[1] Zhao, W.,Maffa, S., & Sandstede, B., *Quantifying Patterns and Their Transitions in Spatially Extended Systems*, in prep.
[2] Swift, J. &d Hohenberg, P. C., *Hydrodynamic fluctuations at the convective instability*, Phys. Rev. A.
[3] Barkley, D. *Chemical Waves and Patterns*, Springer Netherlands.